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Analytic Solution for a Class of Discrete-Time Riccati Equations Arising in Nash Games

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1. INTRODUCTION

When dealing with economic or engineering systems, the control agents or decision makers have rarely the same objective function. An equilibrium solution must be sought and a Nash strategy is a first natural choice. An open-loop Nash strategy for a discrete-time linear quadratic game is considered here. Consider a deterministic two-player linear quadratic non-zero sum discrete time game:

$$x(k+1) = Ax(k) + B_1 u_1(k) + B_1 u_2(k) \tag{1}$$

where A is an invertible matrix in $R^{n\times n}$, $x(k) \in R^n$, $u_1(k) \in R^{r_1}$, $u_2(k) \in R^{r_2}$, for $i = 1, 2$. The performance index associated with player "i" $(i = 1, 2)$ is

$$J_i = \frac{1}{2}\left\{ x^T(N)K_{if}x(N) + \sum_{k=0}^{N-1}\left[x^T(k)Q_i x(k) + u_1^T(k)R_{i1}u_1(k) + u_2^T(k)R_{i2}u_2(k)\right]\right\} \tag{2}$$

where all weighting matrices are symmetric and R_{i1}, for $i = 1, 2$, are positive definite.

If an open-loop Nash solution is sought, each player optimizes his cost function under the assumption the $\frac{\partial u_1(k)}{\partial x(k)} = \frac{\partial u_2(k)}{\partial x(k)} = 0$, [8]. Therefore, the necessary conditions to be satisfied are ([7], [2]):

$$u_i(k) = -R_{ii}^{-1}B_i^T\Psi_i(k+1), \qquad i = 1, 2 \tag{3}$$

and the costate vectore $\Psi_i(k)$, $i = 1, 2$, must verify

$$\Psi_i(k) = Q_i x(k) + A^T\Psi_i(k+1), \quad \Psi_i(N) = K_{if}x(N), \qquad i = 1, 2 \tag{4}$$

When introducing the linear transformations:

$$\Psi_i(k) = K_i(k)x(k), \qquad i = 1, 2 \tag{5}$$

and using (3), system (1) becomes

$$x(k+1) = \{I + S_1K_1(k+1) + S_2K_2(k+1)\}^{-1}Ax(k) \tag{6}$$

with $S_i = B_iR_{ii}^{-1}B_i^T$, for $i = 1, 2$, I being the identity matrix of order n.

Coupled Riccati matrix equations are obtained by substituting (5) and (6) into (4), i.e.

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$$K_1(k) = Q_1 + A^T K_1(k+1)\{I + S_1 K_1(k+1) + S_2 K_2(k+1)\}^{-1} A, \quad K_1(N) = K_{1f},$$

$$K_2(k) = Q_2 + A^T K_2(k+1)\{I + S_1 K_1(k+1) + S_2 K_2(k+1)\}^{-1} A, \quad K_2(N) = K_{2f} \quad (7)$$

These equations can be solved recursively from the terminal conditions since for $k = N$ they are decoupled [4]. However the iterative solution has some inconvenients such as rounding error accumulation, and that if the boundary conditions are changed then the recursive algorithm has to be started all over again. Also, the recursive method needs the storage of $K_1(k), K_2(k)$, along the whole horizon and it involves a considerable memory storage when N is large. In a recent paper [1], an explicit solution of the problem (7) for the case where matrices S_i are proportional, i.e., $S_2 = \alpha S_1$, where α is a scalal, is suggested. The aim of this paper is to find an analytic closed form solution of (7) for the case where there is a matrix L such that

$$S_2 + S_1 L = 0, \qquad L A^T = A^T L \quad (8)$$

Note that when $S_2 = \alpha S_1$, where α is a scalar, taking $L = \alpha I$, the system (8) is satisfied.

If A and B are matrices, we denote by $A \otimes B$, its tensor product, and if M, N and P are matrices of suitable dimensions, then using the column lemma [5, p.410], we get

$$vec(MNP) = (P^T \otimes M) vec N \quad (9)$$

If S is a matrix in $R^{n \times m}$, we denote by S^+ the Moore-Penrose of S, and we recall that an account of properties and implementable procedures for computing S^+ may be found in [3].

2. Analytic Closed Form Solution

Let us consider a new index m defined by $m = N - k$, and let us consider the change defined by the equations

$$\hat{x}(m) = x(N-m) = x(k); \quad \hat{\Psi}_1(m) = \Psi_1(N-m) = \Psi_1(k); \hat{\Psi}_2(m) = \Psi_2(N-m) = \Psi_2(k) \quad (10)$$

and rearranging and combining equations (3)–(6), one gets that the conditions to be satisfied by an open-loop Nash strategy can be written in the matrix form

$$\begin{bmatrix} \hat{x}(m+1) \\ \hat{\Psi}_1(m+1) \\ \hat{\Psi}_2(m+1) \end{bmatrix} = M \begin{bmatrix} \hat{x}(m) \\ \hat{\Psi}_1(m) \\ \hat{\Psi}_2(m) \end{bmatrix}; \qquad \hat{\Psi}_1(0) = K_{1f}\hat{x}(0), \hat{\Psi}_2(0) = K_{2f}\hat{x}(0) \quad (11)$$

where

$$M = \begin{bmatrix} A^{-1} & A^{-1}S_1 & A^{-1}S_2 \\ Q_1 A^{-1} & A^T + Q_1 A^{-1} S_1 & Q_1 A^{-1} S_2 \\ Q_2 A^{-1} & Q_2 A^{-1} S_1 & A^T + Q_2 A^{-1} S_2 \end{bmatrix} \quad (12)$$

Now, let us consider the change of basis

$$\begin{bmatrix} \hat{x} \\ \hat{\Psi}_1 \\ \hat{\Psi}_2 \end{bmatrix} = T \begin{bmatrix} \hat{x} \\ \hat{w} \\ \hat{\Psi}_2 \end{bmatrix}; \qquad T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & L \\ 0 & 0 & I \end{bmatrix} \quad (13)$$

for an appropriated matrix L in $R^{n\times n}$ to be determined. Thus problem (11) is equivalent to the following one

$$\begin{bmatrix} \hat{x}(m+1) \\ \hat{w}(m+1) \\ \hat{\Psi}_2(m+1) \end{bmatrix} =$$

$$\begin{bmatrix} A^{-1} & A^{-1}S_1 & A^{-1}(S_2+S_1L) \\ (Q_1-LQ_2)A^{-1} & A^T+(Q_1-LQ_2)A^{-1}S_1 & A^TL-LA^T+(Q_1-LQ_2)A^{-1}(S_2+S_1L \\ Q_2A^{-1} & Q_2A^{-1}S_1 & A^T+Q_2A^{-1}(S_2+S_1L) \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \hat{x}(m) \\ \hat{w}(m) \\ \hat{\Psi}_2(m) \end{bmatrix} \tag{14}$$

$$\hat{\Psi}_1(0) = K_{1f}\hat{x}(0); \quad \hat{w}(0) = (K_{1f} - LK_{2f})\hat{x}(0) \tag{15}$$

Note that if L is a matrix satisfying (8), then the coefficient matrix of (14) is reduced to the block triangular form

$$\begin{bmatrix} \hat{x}(m+1) \\ \hat{w}(m+1) \\ \hat{\Psi}_2(m+1) \end{bmatrix} = \begin{bmatrix} & & 0 \\ & V & 0 \\ Q_2A^{-1} & Q_2A^{-1}S_1 & A^T \end{bmatrix} \begin{bmatrix} \hat{x}(m) \\ \hat{w}(m) \\ \hat{\Psi}_2(m) \end{bmatrix} \tag{16}$$

$$\hat{\Psi}_1(0) = K_{1f}\hat{x}(0); \quad \hat{w}(0) = (K_{1f} - LK_{2f})\hat{x}(0) \tag{17}$$

where

$$V = \begin{bmatrix} A^{-1} & A^{-1}S_1 \\ (Q_1-LQ_2)A^{-1} & A^T+(Q_1-LQ_2)A^{-1}S_1 \end{bmatrix} \tag{18}$$

Solving system (16) we have

$$\begin{bmatrix} \hat{x}(m) \\ \hat{w}(m) \end{bmatrix} = V^m \begin{bmatrix} \hat{x}(0) \\ \hat{w}(0) \end{bmatrix} = V^m \begin{bmatrix} I \\ K_{1f} - LK_{2f} \end{bmatrix} = G(m), \qquad m \geq 0 \tag{19}$$

$$\hat{\Psi}_2(m) = \left\{ (A^t)^m K_{2f} + \sum_{j=0}^{m-1} (A^T)^{m-j-1} Q_2 A^{-1} [I, S_1] V^j \begin{bmatrix} I \\ K_{1f} - LK_{2f} \end{bmatrix} \right\} \hat{x}(0), \quad m \geq 1 \tag{20}$$

Let us assume the invertibility of $[I,0]G(m)$, then from (19) we have

$$\hat{x}(0) = \{[I,0]G(m)\}^{-1}\hat{x}(m) \tag{21}$$

Hence and from (19), (20), it follows that

$$\hat{w}(m) = [0,I]G(m)\hat{x}(0) = \{[0,I]G(m)\}\{[I,0]G(m)\}^{-1}\hat{x}(m)$$

$$\hat{\Psi}_2(m) = \left\{ (A^T)^m K_{2f} + \sum_{j=0}^{m-1} (A^T)^{m-j-1} Q_2 A^{-1} [I, S_1] G(j) \right\} \{[I,0]G(m)\}^{-1} \hat{x}(m) \tag{22}$$

$$\hat{\Psi}_1(m) = \hat{w}(m) + L\hat{\Psi}_2(m) \tag{23}$$

From (22) it follows that

$$\hat{\Psi}_2(m) = \left\{ (A^T)^m K_{2f} + \sum_{j=0}^{m-1} (A^T)^{m-j-1} Q_2 A^{-1} [I, S_1] G(j) \right\} \{[I,0]G(m)\}^{-1} \hat{x}(m) \tag{24}$$

Now, from (23) and (24) we have

$$\hat{\Psi}_1(m) = \left\{\{[0, I]G(m)\}\{[I, 0]G(m)\}^{-1} + L\hat{\Psi}_2(m)\right\}\hat{x}(m) \tag{25}$$

Hence and from (5), (10), we have

$$K_2(k) = \left\{(A^T)^{N-k}K_{2f} + \sum_{j=0}^{N-k-1}(A^T)^{N-k-j-1}Q_2A^{-1}[I, S_1]G(j)\right\}\{[I, 0]G(N-k)\}^{-1} \tag{26}$$

$$K_1(k) = \{[0, I]G(N-k)\}\{[I, 0]G(N-k)\}^{-1} + LK_2(k) \tag{27}$$

Thus under the existence of a solution L of the algebraic system (8) and the invertibility of $[I, 0]G(N-k)$, where $G(m)$ is defined by the right hand side of (19), expressions (26) and (27) provide the solution of the coupled Riccati system (7).

Now we characterize in terms of data the existence of a solution L of system (8). If we apply the column lemma and (9) to each equation of system (8), we obtain the equivalent algebraic system

$$C \; vec \; L = vec \; [0, -S_2] \tag{28}$$

where

$$C = \begin{bmatrix} I \otimes A^T - A \otimes I \\ I \otimes S_1 \end{bmatrix} \tag{29}$$

Now, from th.2.3.2 of [6], the algebraic system (28) – (29) is compatible, if and only if,

$$CC^+[0, -S_2] = vec \; [0, -S_2] \tag{30}$$

and in this case, the general solution of (28) is given by

$$vec \; L = C^+ vec \; [0, -S_2] + (I_{n^2} - C^+C)Z \tag{31}$$

where I_{n^2} denotes the identity matrix in $R^{n^2 x n^2}$ and Z is an arbitrary vector in R^{n^2}. Thus taking into account that the computation of the matrix C^+ is straightforward, the condition (30) is easy to verify and the computation of a solution L of (8) is an easy matter. We summarize the result in the following theorem:

THEOREM 1. *Let us consider the coupled Riccati system* (7) *and let us assume that matrices* A, S_1 *and* S_2 *satisfy the condition* (30). *Let* L *be a solution of system* (8) *and let* $G(m)$ *be defined by* (19) *for* $0 \le m \le N$. *If* k *is an integer such that* $0 \le k \le N$ *and* $[I, 0]G(N-k)$ *is nonsingular, then the solution* $K_1(k), K_2(k)$ *of* (7) *is given by* (27) *and* (26) *respectively.*

Remark: Note that to compute the solution $K_1(k), K_2(k)$, it is necessary to make some multiplications, additions and one matrix inversion with time dependent matrices. These operations made can be done quite simply on a computer using an algebraic manipulation language such as REDUCE or MACSYMA. These algebraic manipulation languages are used only in the final steps to make operations involving time-dependent matrices.

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